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The Fronsdal massless equations for integer spin in Duffin–Kemmer form

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Abstract. The Fronsdal integer spin equations for zero mass are put into first-order Duffin–Kemmer form. The minimal polynomials of the coefficient matrices are determined and compared with the massive case. It is also noted that the various antisymmetric tensor gauge field theories of recent interest are included in a general massless Duffin–Kemmer formalism of Harish-Chandra.

1. Introduction

There has been increasing interest in massless, high-spin field theories in recent years, encouraged in part by the needs of extended supergravity (Fronsdal 1978, Christensen and Duff 1979, Fang and Fronsdal 1978, Berends *et al* 1979, Berends and Van Reisen 1980, Curtwright 1979, 1980, De Wit and Freedman 1980, Fronsdal and Hata 1980, Aragone 1981). A number of different tensor–spinor formulations of such field theories have been given and discussed, but perhaps the most commonly used form is that of Fang and Fronsdal (1978), which is essentially the massless limit of the general spin Fierz–Pauli equations as formulated by Singh and Hagen (1974).

There has also been much interest in lower-spin ($\leqslant 1$) theories which use higher-rank tensors, particularly antisymmetric tensors, because of the occurrence of such tensor fields in extended supergravity theories (Hagen 1979, Sezgin and Van Nieuwenhuizen 1980, Aurilia and Takahashi 1981, Duff 1981, Deser and Witten 1981, Deser *et al* 1981, Townsend 1981, Obhukhov 1982, Van Nieuwenhuizen 1982). Finally, there is the recent use of linear coupling of higher-rank gauge fields to generate massive field theories by the 'gauge-mixing mechanism' (Hagen 1979, Aurilia and Takahashi 1981, Govindarajan 1982).

Despite repeated efforts (Berends *et al* 1980, Aragone and Deser 1979, Aragone 1981) no consistent interaction coupling has been achieved for massless fields for helicity greater than 2, and the suspicion has grown that no such coupling is possible. This may be, but high-spin massless theories have received comparatively little attention compared with their massive counterparts, and further study of the available possibilities seems worthwhile.

A useful general approach to the massive high-spin theory is to write the field equations in the so called Duffin–Kemmer matrix differential form:

$$\mathbf{L}_\mu \partial^\mu + \mathrm{i}\chi \mathbf{I}\psi = 0 \qquad (1.1)$$

where $\mathbf{L}_\mu$ are square matrices, χ a real number, $\mathbf{I}$ a unit matrix and ψ a column vector of the field components (Kemmer 1939, Gel'fand *et al* 1963, Takahashi 1969, Velo and Zwanzinger 1971, Cox 1978, 1981). Such equations have the advantage that they are independent of the tensor–spinor form of the fields used to represent the various Lorentz group irreps in ψ, and they allow a simple and systematic analysis of the mass–spin spectra by study of the eigenvalues of $\mathbf{L}_0$.

In the massive case there are two distinct approaches to such equations as (1.1), by the study of the abstract tensor algebra generated by the $\mathbf{L}_\mu$; or by explicit construction of the $\mathbf{L}_\mu$ matrices from the conditions of covariance, Lagrangian origin and mass–spin spectra. The prototype for the former approach is the original Duffin–Kemmer theory of spin 0 and 1, for which the $\mathbf{L}_\mu$ algebra is given by

$$\mathbf{L}_\mu \mathbf{L}_\nu \mathbf{L}_\rho + \mathbf{L}_\rho \mathbf{L}_\nu \mathbf{L}_\mu = g_{\mu\nu} \mathbf{L}_\rho + g_{\rho\nu} \mathbf{L}_\mu \tag{1.2}$$

or, with

$$P = p^\mu \mathbf{L}_\mu$$

$$P(P^2 - p^2) = 0.$$

By either approach, the important aspect of (1.1) is the minimal equation of $\mathbf{L}_0$, since it is this which largely determines the mass–spin spectra and the complexity of the constraints in the theory. For higher spin and unique mass the algebra of the $\mathbf{L}_\mu$ is much more complicated than (1.2).

Despite the large amount of work done on the massive Duffin–Kemmer type of equation (1.1), there has been very little published on massless theories in a similar form. For such theories the scalar 'mass matrix' $\chi\mathbf{I}$ must be replaced by a singular matrix $\mathbf{M}$, and the massive form (1.1) has to be rewritten slightly if it is to have a non-trivial massless limit. For example Harish-Chandra (1946) showed that the electromagnetic field equations could be written in the form

$$\mathrm{i}\beta_\mu \partial^\mu \psi + \mathbf{M}\psi = 0 \tag{1.3}$$

where the β_μ satisfy the Duffin–Kemmer algebra (1.2), and further

$$\mathbf{M}^2 = \mathbf{M} \tag{1.4}$$

$$\mathbf{M}\beta_\mu + \beta_\mu \mathbf{M} = \beta_\mu. \tag{1.5}$$

It is easily verified that equation (1.3) is invariant under the gauge transformation

$$\psi \to \psi' = \psi + (\mathbf{I} - \mathbf{M})\xi \tag{1.6}$$

where ξ is any solution of the equation

$$\mathrm{i}\beta_\mu \partial_\mu \xi = 0. \tag{1.7}$$

This formulation of Harish-Chandra was extended independently by Okubo and Tosa (1979) to non-Abelian Yang–Mills theories, and also to Einstein gravity, although in the latter case they did not give the β algebra.

It is interesting to note that Harish-Chandra (1946) proved that the theory of (1.3), (1.4) and (1.5), with the Duffin–Kemmer algebra (1.2), in fact includes a sequence of four massless gauge theories depending on the choice of β_μ representation and the choice of $\mathbf{M}$. Thus Harish-Chandra showed that for each of the two non-trivial irreducible representations of the β_μ algebra, there are just two matrices, $\mathbf{M}$ and $\mathbf{I} - \mathbf{M}$, satisfying (1.4) and (1.5). In the ten-dimensional β representation, if one chooses $\mathbf{M}$

to satisfy

$$\mathbf{M} = \tfrac{1}{2}\beta_\mu(\mathbf{I}-\mathbf{M})\beta^\mu$$

then (1.3) is equivalent to the Maxwell equations for a vector field. Then replacing $\mathbf{M}$ by $\mathbf{I}-\mathbf{M}$ in (1.3) yields a system equivalent to the gauge theory of a second-rank antisymmetric potential $A_{\mu\nu}$:

$$\partial_\mu A^{\nu\mu} = A^\nu$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = 0$$

with gauge invariance under

$$\delta A^\mu = 0 \qquad \delta A^{\mu\nu} = \varepsilon^{\mu\nu\rho\alpha}\partial_\rho A_\alpha$$

(Deser and Witten 1981). For the five-dimensional β representation we take $\mathbf{M}$ to satisfy

$$\mathbf{M} = \tfrac{1}{4}\beta_\mu(\mathbf{I}-\mathbf{M})\beta^\mu.$$

The resulting theory is equivalent to the gauge theory of a third-rank antisymmetric linear potential $A_{\mu\nu\rho}$, which propagates no degrees of freedom (Townsend 1981). Replacing $\mathbf{M}$ by $\mathbf{I}-\mathbf{M}$ in (1.3) gives the usual massless scalar theory. Thus, considered algebraically, the system (1.2) to (1.5) provides a succinct unified treatment of the complete set of antisymmetric linear gauge theories which have been studied recently.

There have been other attempts to reformulate specific massless theories in the typical form (1.3) with or without (1.4) and (1.5), but these are mainly limited to spin $\frac{1}{2}$ (Sen Gupta 1967, Santhanam and Chandrasekaran 1969, Samiullah and Mansour 1981) or spin 2 (Brulin and Hjalmars 1964). The only attempt at a general study of equations of the type (1.3) seems to have been in the unpublished thesis of Kwoh (1970), who gave sufficient conditions for such an equation to have massless (and massive) solutions and used these to obtain a theory describing a scalar field having both massless and massive modes (Kwoh 1970). Massive Duffin–Kemmer-type-theories with a singular mass matrix have received very little study. Theories obtained by the gauge-mixing mechanism of Aurilia and Takahashi (1981) are of this type.

In this paper we study some of the properties of massless theories in Duffin–Kemmer form. In particular we reformulate the massless Fronsdal equations for integer spin in the Duffin–Kemmer-type form (1.3), and derive the minimal polynomial of the β_0 matrix. The result is a generalisation of the Harish-Chandra theory in the sense that the relations (1.4) and (1.5) are satisfied, but the β algebra of the massless theory is more complicated than that of the corresponding massive theory. The method we use is to write the Singh–Hagen equation in first-order form in such a way that the massless limit may be taken in a straightforward way, and to arrive at the Fronsdal equations via this limit. The minimal polynomial of β_0 for the Singh–Hagen equations is known, and is used to deduce that of the Fronsdal equations.

In § 2 we review the general theory of massive and massless Duffin–Kemmer-type equations and in § 3 derive the minimal polynomials for the Fronsdal equations.

2. Massive and massless first-order equations

Consider a system of manifestly Lorentz covariant first-order field equations, describing a field with arbitrary mass–spin spectra, which may include massive or massless modes, or both. Any system of higher-order equations can always be written in this form, by introduction of a sufficient number of new variables. The most general matrix form of such a system can be written

$$(\boldsymbol{\Gamma}_{\mu}\partial^{\mu}+\mathrm{i}\boldsymbol{\mathcal{M}})\psi=0 \tag{2.1}$$

where $\boldsymbol{\Gamma}_{\mu}$, $\boldsymbol{\mathcal{M}}$ are matrices, and ψ a column vector representing the fields. Now $\boldsymbol{\Gamma}_{\mu}$ and $\boldsymbol{\mathcal{M}}$ do not necessarily have to be square, particularly in the case of massless theories. However, if the equations are to derive from a real Lagrangian then it is desirable that the equations transform in the same way as the fields, and so we will take the $\boldsymbol{\Gamma}_{\mu}$, $\boldsymbol{\mathcal{M}}$ to be square, regardless of the mass–spin spectra.

We assume that the field ψ lies in the respresentation space $\mathscr{R}$ of a (reducible) representation of the proper Lorentz group ($\mathscr{L}_{\mathrm{P}}$) and under Lorentz transformations transforms according to

$$x^{\mu}\to x^{\mu'}=a^{\mu}{}_{\nu}x^{\nu} \qquad \psi\to\psi'=\psi'(x')=\mathbf{T}_{a}\psi(x). \tag{2.2}$$

We assume that the representation space $\mathscr{R}$ is fully reducible with respect to $\mathscr{L}_{\mathrm{P}}$, so that ψ can be written as a direct sum of components transforming irreducibly under $\mathscr{L}_{\mathrm{P}}$. Manifest covariance of (2.1) under the transformation (2.2) demands that the $\boldsymbol{\Gamma}_{\mu}$, $\boldsymbol{\mathcal{M}}$ matrices satisfy

$$\mathbf{T}^{-1}\boldsymbol{\Gamma}_{\mu}\mathbf{T}=a_{\mu}{}^{\nu}\boldsymbol{\Gamma}_{\nu} \tag{2.3a}$$

$$\mathbf{T}^{-1}\boldsymbol{\mathcal{M}}\mathbf{T}=\boldsymbol{\mathcal{M}}. \tag{2.3b}$$

If we further require that (2.1) come from a real invariant Lagrangian, then there must exist an invariant hermitising matrix $\boldsymbol{\Lambda}$ such that

$$\mathbf{T}^{\dagger}\boldsymbol{\Lambda}\mathbf{T}=\boldsymbol{\Lambda} \tag{2.4a}$$

$$\boldsymbol{\Gamma}_{\mu}^{\dagger}\boldsymbol{\Lambda}=\boldsymbol{\Lambda}\boldsymbol{\Gamma}_{\mu} \tag{2.4b}$$

$$\boldsymbol{\mathcal{M}}^{\dagger}\boldsymbol{\Lambda}=\boldsymbol{\Lambda}\boldsymbol{\mathcal{M}}. \tag{2.4c}$$

Then (2.1) can be obtained from the Lagrangian

$$\mathbf{L}=\mathrm{i}^{-1}\psi^{\dagger}\boldsymbol{\Lambda}(\boldsymbol{\Gamma}_{\mu}\partial^{\mu}+\mathrm{i}\boldsymbol{\mathcal{M}})\psi \tag{2.5}$$

by varying ψ or $\psi^{\dagger}$ independently. Note that for the existence of a non-degenerate $\boldsymbol{\Lambda}$ it is necessary that ψ be a self-conjugate representation of $\mathscr{L}_{\mathrm{P}}$, which we henceforth assume (Gel'Fand *et al* 1963).

Two distinct types of theories now arise, depending on whether $\boldsymbol{\mathcal{M}}$ is non-singular or singular. In the first case only massive states may occur, while the latter will accommodate massless and massive modes. Although we are mainly concerned with massless theories we will need first to review the massive case.

2.1. $\boldsymbol{\mathcal{M}}$ non-singular

In this case we can always multiply through by $\chi\boldsymbol{\mathcal{M}}^{-1}$, χ a real number, and put $\mathbf{L}_{\mu}=\chi\boldsymbol{\mathcal{M}}^{-1}\boldsymbol{\Gamma}_{\mu}$, $\eta=\chi^{-1}\boldsymbol{\Lambda}\boldsymbol{\mathcal{M}}$ and consider the equation in the form

$$(\mathbf{L}_{\mu}\partial^{\mu}+\mathrm{i}\chi\mathbf{I})\psi=0 \tag{2.6}$$

where, from (2.3) and (2.4), η is Hermitian and

$$\mathbf{T}^{-1}\mathbf{L}_\mu\mathbf{T} = a_\mu{}^\nu\mathbf{L}_\nu \tag{2.7a}$$

$$\eta\mathbf{L}_\mu^\dagger = \mathbf{L}_\mu\eta. \tag{2.7b}$$

The theory of equations (2.6) and (2.7) is thoroughly understood (Gel'Fand *et al* 1963, Cox 1978, 1981) and we summarise only the points we will need.

If we look for solutions to (2.6) of the form $\psi = \phi\, \mathrm{e}^{\mathrm{i}px}$, we obtain

$$(p\mathbf{L} + \chi\mathbf{I})\phi = 0 \tag{2.8}$$

for which non-trivial solutions exist only if

$$|p\mathbf{L} + \chi\mathbf{I}| = 0, \tag{2.9}$$

that is if the non-zero number χ is an eigenvalue of the matrix $\mathbf{P} = p\mathbf{L}$. Relativistic covariance implies that the eigenvalues of $\mathbf{P}$ are either zero or homogeneous functions in p^2 (Bhabha 1949, Udgaonkar 1952) from which it follows that $p^2 = 0$ cannot be a root of (2.9) and so theories for which $\mathcal{M}$ is non-singular cannot describe massless particles.

For massive fields we may use a similarity transformation with $\mathbf{T}$ to convert (2.8) to rest-frame form

$$(-p_0\mathbf{L}_0 + \chi\mathbf{I})\phi = 0$$

from which we see that the rest-mass values are given by $p_0 = \chi/\lambda$ where λ is a non-zero eigenvalue of $\mathbf{L}_0$. Further, once $\mathbf{L}_0$ is known, the $\mathbf{L}_i$ can be obtained from a Lorentz transformation.

As is well known (Gel'Fand *et al* 1963), $\mathbf{L}_0$ commutes with the generator of space rotations and so may be partitioned into block-diagonal form by arranging all basis vectors in $\mathcal{R}$ with the same total spin together into the so-called spin blocks (Gel'Fand *et al* 1963, Cox 1974). The 'elements' of the r block are $(2r+1)\times(2r+1)$ scalar matrices and so will be treated as numbers. This being understood, the dimension of the r block is determined by the number of Lorentz irreps in $\mathcal{R}$ which contain spin r. Suppose the irreps $\tau_1, \tau_2, \ldots, \tau_n$ occur in the r block $\mathbf{L}_0^{(r)}$. Then the ij element of $\mathbf{L}_0^{(r)}$ is non-zero only if the irreps $\tau_i \sim (k_i, l_i)$, $\tau_j \sim (k_j, l_j)$ are *linked*, that is

$$k_j = k_i \pm \tfrac{1}{2} \qquad l_j = l_i \pm \tfrac{1}{2}.$$

At most four different irreps can be linked to any given irrep τ_i. This linkage property motivates a useful graphical representation of the r blocks, or indeed the complete $\mathbf{L}_0$ of the theory (Cox 1974, 1978, 1981) in which the Lorentz irreps of the theory are taken as vertices and the non-zero elements of the s blocks (or $\mathbf{L}_0$) as the directed edges of the graph, connecting appropriate vertices. Particularly, if the $\mathcal{L}_\mathrm{p}$ irreps each occur only once, the result is a planar linear graph of very simple structure.

A non-zero eigenvalue λ of $\mathbf{L}_0^{(r)}$ corresponds to a field mode with mass χ/λ and total spin r. So the eigenvalue spectra of the $\mathbf{L}_0^{(r)}$ are very important, and graphical methods have been developed to assist in the study of these (Cox 1974, 1978, 1981). The elements of $\mathbf{L}_0^{(r)}$ are of the general form $\rho(k_i, l_i, r)C^{\tau_i\tau_j}$ where $C^{\tau_i\tau_j}$ is a complex parameter. The requirements of (2.7b) and of space reflection covariance relate $C^{\tau_j\tau_i}$, $C^{\tau_i^c\tau_j^c}$ and $C^{\tau_j^c\tau_i^c}$ to $C^{\tau_i\tau_j}$ (τ_i^c denotes the conjugate representation to τ_i), and further restrictions on the $C^{\tau_i\tau_j}$ result from any mass–spin spectra imposed. For a theory based on (2.6) and (2.7) describing a particle–antiparticle field with unique mass χ

and unique spin s, we require the s block $\mathbf{L}_0^{(s)}$ to have eigenvalues 0, ± 1, and a minimal polynomial of the form

$$m(\lambda) = \lambda^{q_s}(\lambda^2 - 1)$$

while all other blocks, $\mathbf{L}_0^{(j)}$, have to be nilpotent:

$$(\mathbf{L}_0^{(j)})^{q_j} = 0 \qquad j \neq s.$$

Unfortunately, widely studied though it is, the massive form (2.6) is not suitable for our present purposes, because it does not have a non-trivial massless ($\chi \to 0$) limit. On putting $\chi = 0$ we obtain the completely covariantly reducible system

$$\mathbf{L}\partial\psi = 0$$

which splits up into a number of separately covariant systems of equations. As we shall see for the Fronsdal massless equations, to express the massive theory in a form suitable for taking the massless limit we must utilise the original form (2.5) in which $\mathcal{M}$ is non-singular, but is not a scalar matrix. Fortunately, we find that in the case of the Singh–Hagen equations this can be done without destroying the simple form of the $\mathbf{L}_0$ algebra of this theory.

2.2. $\mathcal{M}$ singular (*but non-zero*)

In this case we have to deal with (2.1), (2.3) and (2.4). From (2.3), $\mathcal{M}$ commutes with the representation matrix $\mathbf{T}$, and as this is a direct sum of Lorentz irreps, $\mathcal{M}$ can be written in the block-diagonal form

$$\mathcal{M} = \begin{bmatrix} \chi_1 & & \cdots & & \\ & \chi_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & \cdots & & \chi_n \end{bmatrix}$$

where χ_i is a scalar matrix corresponding to the $\mathcal{L}_p$ irrep τ_i, all other entries being zero. Some of the χ_i will also be zero, and the rest can all be rescaled to unity without loss of generality, so that $\mathcal{M}$ can be rearranged covariantly in the form

$$\mathcal{M} = \mathbf{M} = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{bmatrix}$$

satisfying $\mathbf{M}^2 = \mathbf{M}$.

It is important to note that in general there is no *a priori* relation between $\mathbf{\Gamma}_\mu$ and $\mathbf{M}$. Further, the nice connection between mass–spin spectra and the $\mathbf{\Gamma}_\mu$ algebra is now weakened considerably and requires reappraisal. We can derive some general points from (2.1), (2.3) and (2.4). For example, since the charge density of the Lagrangian density, (2.5), is proportional to $\psi^+ \mathbf{M}\psi$, physical states cannot lie in the kernel of $\mathbf{M}$ because such states would not contribute to the charge density. If we assume a plane-wave solution $\psi = \phi e^{ipx}$ to (2.1) we obtain (with $\mathcal{M} = \mathbf{M}$)

$$(\mathbf{\Gamma}p + \mathbf{M})\phi = 0.$$

For solutions with time-like p_μ we may transform to a rest frame and reduce the problem to the form

$$(-\mathbf{\Gamma}_c\, p_0 + \mathbf{M})\phi = 0.$$

If there are any non-trivial solutions to this equation then these would represent massive field modes. For massless states (light-like p_μ) we cannot go to a rest frame and can at best choose some light-like p_μ, say $(0, 0, p_0, p_0)$ and consider the problem

$$[p_0(\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_0) + \mathbf{M}]\phi = 0.$$

Massless states will only exist if this has a non-trivial solution. We can say little more without being more specific about the forms of $\boldsymbol{\Gamma}_\mu$ and $\mathbf{M}$, and so we now turn to the example provided by the Fronsdal equations for integer spin.

3. The Fronsdal massless equations for integer spin

The Fronsdal equations are obtained by putting $m^2 = 0$ in the Singh–Hagen equations for arbitrary spin, massive fields (Singh and Hagen 1974). For spin $s > 3$ the Singh–Hagen equations in first-order form involve two sets of tensor fields, $\phi^{(p)}_{\mu_1\ldots\mu_p}$, $p = 0$, $1, 2, \ldots, (s-2), s$, which are symmetric and traceless and $H^{(p)}_{\mu_1\ldots\mu_{p-1},\mu\alpha}$ $(p = s, s-2, s-3, \ldots, 2)$, $H^{(1)}_{\mu\alpha}$, $H^{(0)}_\alpha$, H, having the following properties.

(i) $H^{(p)}_{\mu_1\ldots\mu_{p-1},\mu\alpha}$ is antisymmetric in μ and α and symmetric traceless in the remaining indices.

(ii) $\varepsilon^{\mu_1\mu\alpha\beta}H^{(p)}_{\mu_1\ldots\mu_{p-1},\mu\alpha} = 0$, $p = s, (s-3), \ldots, 2$.

(iii) $g^{\mu\mu_1}H^{(s-2)}_{\mu_1\ldots\mu_{s-3},\mu\alpha} = 0$.

The Lorentz irreps involved are

$$\phi^{(p)}_{\mu_1\ldots\mu_p} \sim \mathscr{D}(\tfrac{1}{2}p, \tfrac{1}{2}p), \qquad p = (0, 1, 2, \ldots, (s-1), s)$$

$$H^{(s)}_{\mu_1\ldots\mu_{s-1},\mu\alpha} \sim \mathscr{D}[\tfrac{1}{2}(s+1), \tfrac{1}{2}(s-1)] \oplus \mathscr{D}[\tfrac{1}{2}(s-1), \tfrac{1}{2}(s+1)] \oplus \mathscr{D}[\tfrac{1}{2}(s-1), \tfrac{1}{2}(s-1)]$$

$$H^{(s-2)}_{\mu_1\ldots\mu_{s-3},\mu\alpha} \sim \mathscr{D}[\tfrac{1}{2}(s-1, \tfrac{1}{2}(s-3)] \oplus \mathscr{D}[\tfrac{1}{2}(s-3), \tfrac{1}{2}(s-1)]$$

$$H^{(p)}_{\mu_1\ldots\mu_{p-1},\mu\alpha} \sim \mathscr{D}(1+\tfrac{1}{2}p, \tfrac{1}{2}p) \oplus \mathscr{D}(\tfrac{1}{2}p, 1+\tfrac{1}{2}p) \oplus \mathscr{D}(\tfrac{1}{2}p, \tfrac{1}{2}p) \qquad p = 0, 1, 2, \ldots (s-4)$$

$$H^{(0)}_\mu \sim \mathscr{D}(\tfrac{1}{2}, \tfrac{1}{2})$$

($H^{(1)}_{\mu\alpha}$ and H are included together under the case $p = 0$).

The notation $\{\mathbf{T}^{(p)}_{\mu_1\ldots\mu_p}\}_{\mathrm{ST}}$ will be used to denote the symmetric traceless part of a tensor, while $\{\mathbf{T}^{(p)}_{\mu_1,\ldots,\mu_p}\}_{\mathrm{A}}$ denotes that part satisfying the conditions (i) and (ii) above. With these notations and conventions, the Singh–Hagen massive equations are

$$H^{(s)}_{\mu_1\ldots\mu_{s-1},\mu\alpha} = 2\{\partial_\alpha\phi^{(s)}_{\mu\mu_1\ldots\mu_{s-1}} + (s-1)ag_{\alpha\mu_1}(\partial\phi^{(s)})_{\mu\mu_2\ldots\mu_{s-1}} + (s-1)^2bg_{\mu\mu_1}\partial_\alpha\phi^{(s-2)}_{\mu_2\ldots\mu_{s-1}}\}_{\mathrm{A}} \tag{3.1a}$$

$$H^{(s-2)}_{\mu_1\ldots\mu_{s-3},\mu\alpha} = 2\{\partial_\alpha\phi^{(s-2)}_{\mu\mu_1\ldots\mu_{s-3}}\}_{\mathrm{A}} \tag{3.1b}$$

$$H^{(s-q)}_{\mu_1\ldots\mu_{(s-q-1)},\mu\alpha} = 2\{\partial_\alpha\phi^{(s-q)}_{\mu\mu_1\ldots\mu_{(s-q-1)}} + (s-q-1)\lambda_q g_{\alpha\mu_1}(\partial\phi^{(s-q)})_{\mu\mu_2\ldots\mu_{(s-q-1)}}\}_{\mathrm{A}}$$
$$3 \leq q \leq (s-2) \tag{3.1c}$$

$$H^{(1)}_{\mu\alpha} = \partial_\alpha\phi^{(1)}_\mu - \partial_\mu\phi^{(1)}_\alpha \qquad H = \partial\phi^{(1)} \qquad H^{(0)}_\alpha = \partial_\alpha\phi^{(0)} \tag{3.1d}$$

$$\{\partial^\alpha H^{(s)}_{\mu_1\ldots\mu_{(s-1)},\mu_s\alpha} + \lambda\,\partial_{\mu_1}\bar{H}^{(s)}_{\mu_2\ldots\mu_s}\}_{\mathrm{ST}} - \chi^2\phi^{(s)}_{\mu_1\ldots\mu_s} = 0 \tag{3.2a}$$

$$[1+a(s+1)]^{-1}\partial^\alpha\bar{H}^{(s)}_{\mu_1\ldots\mu_{(s-2)},\alpha} - [(s-2)/(2s-1)]\{\partial^\alpha H^{(s-2)}_{\mu_1\ldots\mu_{(s-3)},\mu_{(s-2)}\alpha}\}_{\mathrm{ST}}$$
$$- a_2\chi^2\phi^{(s-2)}_{\mu_1\ldots\mu_{(s-2)}} + \chi c_2\{\partial_{\mu_1}\phi^{(s-3)}_{\mu_2\ldots\mu_{(s-2)}}\}_{\mathrm{ST}} = 0 \tag{3.2b}$$

$$\{\partial^\alpha H^{(p)}_{\mu_1\ldots\mu_{p-1},\mu\alpha} + A_p \partial_{\mu_1} \bar{H}^{(p)}_{\mu_2\ldots\mu_p}\}_{\mathrm{ST}} - \chi(\partial\phi^{(p+1)})_{\mu_1\ldots\mu_p} + \chi c_{s-p}\{\partial_{\mu_1}\phi^{(p-1)}_{\mu_2\ldots\mu_p}\}_{\mathrm{ST}}$$
$$- a_{s-p}\chi^2\phi^{(p)}_{\mu_1\ldots\mu_p} = 0, \qquad p = (s-3), (s-4), \ldots, 2 \tag{3.2c}$$

$$\partial^\alpha\{H^{(1)}_{\mu\alpha} + \tfrac{6}{5}g_{\mu\alpha}H\} - \chi(\partial\phi^{(2)})_\mu + \chi c_{(s-1)}\partial_\mu\phi^{(0)} - a_{(s-1)}\chi^2\phi^{(1)}_\mu = 0 \tag{3.2d}$$

and

$$\partial^\alpha H^{(1)}_\alpha - a_s\chi^2\phi^{(0)} - \chi(\partial\phi^{(1)}) = 0 \tag{3.2e}$$

where

$$\bar{H}^{(p)}_{\mu_1\ldots\mu_{p-1}} \equiv g^{\mu\nu}H^{(p)}_{\mu_1\ldots\mu_{(p-2)},\nu\mu_{(p-1)}}.$$

The precise forms of the coefficients a, b, c, *etc*, do not concern us here.

Dividing through by the coefficients a_p of the $\phi^{(p)}$ in equations (3.2) and inserting a factor i we can rewrite the above systems in the matrix form

$$(\mathbf{\Gamma}_\mu\partial^\mu + \mathrm{i}\mathcal{M})\psi = 0 \tag{3.3}$$

where

$$\mathbf{\Gamma}_\mu = \begin{bmatrix} 0 & \Gamma^{(12)}_\mu \\ \Gamma^{(21)}_\mu & 0 \end{bmatrix} \qquad \mathcal{M} = \begin{bmatrix} \chi^2 & 0 \\ 0 & I \end{bmatrix} \qquad \psi = \begin{bmatrix} \phi \\ H \end{bmatrix} \tag{3.4}$$

in obvious notation.

We cannot convert this system to one of the form (2.6) because the result would not have a non-singular $\chi \to 0$ limit. We must, therefore, work with the form (3.3). But then we have to determine the $\mathbf{\Gamma}_\mu$ algebra and rework the theory of the Singh–Hagen equations already done in the form (2.6) (Cox 1978).

Converting to the form (2.6) by putting

$$\mathbf{L}_\mu = \chi\mathcal{M}^{-1}\mathbf{\Gamma}_\mu = \chi^{-1}\mathbf{\Gamma}_\mu\mathcal{M} \tag{3.5}$$

we have, if q is even,

$$\mathbf{L}^q_\mu = \mathbf{\Gamma}^q_\mu$$

and if q is odd

$$\mathbf{L}^q_\mu = \chi\mathcal{M}^{-1}\mathbf{\Gamma}^q_\mu.$$

These results imply that, so long as $\chi \neq 0$ $\mathbf{\Gamma}_\mu$ satisfies

$$\mathbf{\Gamma}^q_\mu(\mathbf{\Gamma}^2_\mu - 1) = 0$$

if and only if $\mathbf{L}_\mu$ satisfies the same equation:

$$\mathbf{L}^q_\mu(\mathbf{L}^2_\mu - 1) = 0.$$

The same argument may be repeated, in particular for the individual s blocks $\mathbf{\Gamma}^{(r)}_0$ ($\mathbf{\Gamma}_0$ and $\mathcal{M}$ both commute with the rotation group operator and by inspection may be simultaneously partitioned into s blocks in the same manner as (3.4)). Thus the s blocks of $\mathbf{\Gamma}_0$ obey the same algebra as those of $\mathbf{L}_0$, and if we determine $\mathbf{L}_0$ to satisfy the conditions for a unique mass, χ, spin-s field then $\mathbf{\Gamma}_0$ will likewise satisfy the same conditions. So we can use the results obtained by the standard theory of (2.6) for the Singh–Hagen equations and yet still retain the form (2.1) for taking the $\chi \to 0$ limit.

The graph for $\mathbf{L}_0$ for the integer spin, s, Singh–Hagen equations was given by Cox (1981). The s block has to satisfy

$$\mathbf{L}_0^{(s)}(\mathbf{L}_0^{(s)2}-1)=0$$

as minimal polynomial, while all the other blocks are nilpotent, the maximum degree of nilpotency being (from the 1 block) $2s-1$. From what has been said above, the $\mathbf{\Gamma}_0$ spin blocks satisfy the same conditions.

To obtain the Fronsdal equations we now put $\chi=0$ in (3.2) and (3.3). We observe that in this process the fields $\phi^{(p)}_{\mu_1\ldots\mu_p}$, $p=(s-3)$, $(s-4),\ldots,0$, and $H^{(p)}$, $p=(s-3),\ldots,0$, decouple from the equations and may henceforth be ignored, only the fields $\phi^{(s)}$, $\phi^{(s-2)}$, $H^{(s)}$, $H^{(s-2)}$ remaining, the field equations being

$$H^{(s)}_{\mu_1\ldots\mu_{s-1},\mu\alpha}=2\{\partial_\alpha\phi^{(s)}_{\mu\mu_1\ldots\mu_{(s-1)}}+(s-1)ag_{\alpha\mu_1}(\partial\phi^{(s)})_{\mu\mu_2\ldots\mu_{s-1}}+(s-1)^2bg_{\mu\mu_1}\partial_\alpha\phi^{(s-2)}_{\mu_2\ldots\mu_{(s-1)}}\}_A \tag{3.6a}$$

$$H^{(s-2)}_{\mu_1\ldots\mu_{(s-3)},\mu\alpha}=2\{\partial_\alpha\phi^{(s-2)}_{\mu\mu_1\ldots\mu_{(s-3)}}\}_A \tag{3.6b}$$

$$\{\partial^\alpha H^{(s)}_{\mu_1\ldots\mu_{s-1},\mu_s\alpha}+\lambda\,\partial_{\mu_1}\bar H^{(s)}_{\mu_2\ldots\mu_s}\}_{ST}=0 \tag{3.7a}$$

$$[1+a(s+1)]^{-1}\partial^\alpha\bar H^{(s)}_{\mu_1\ldots\mu_{(s-2)}\alpha}-\left(\frac{s-2}{2s-1}\right)\{\partial^\alpha H^{(s-2)}_{\mu_1\ldots\mu_{(s-3)},\mu_{(s-2)}\alpha}\}_{ST}=0. \tag{3.7b}$$

The second-order massless equations of Fronsdal result by eliminating $H^{(s)}$, $H^{(s-2)}$ from these equations. Retaining the above first-order form we can represent the system as

$$(\beta_\mu\partial^\mu+\mathrm{i}\mathbf{M})\psi=0 \tag{3.8}$$

where

$$\beta_\mu=\begin{bmatrix}0 & \beta^{(12)}_\mu\\ \beta^{(21)}_\mu & 0\end{bmatrix}\qquad \mathbf{M}=\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\qquad \psi=\begin{bmatrix}\phi\\ H\end{bmatrix}. \tag{3.9}$$

We notice that the β_μ and $\mathbf{M}$ satisfy equation (1.5) of the Harish-Chandra first-order form of the Maxwell theory. Also, (3.8) is invariant under a generalised form of the gauge transformation (1.6) and (1.7). Thus, the Fronsdal equations are the natural extension, to high spin, of the Harish-Chandra massless theory except, of course, as we shall see, that the β algebra is more complicated. Further, while the same β algebra serves for both the massive and corresponding massless theory (that is the Duffin–Kemmer algebra (1.2)) in the case of spin 1, this is not the case for higher spin. Also note that (1.5) is not a surprising result—it reflects the general form of the system of equations

$$\begin{aligned}\beta^{(12)}_\mu\partial^\mu H&=0\\ \beta^{(21)}_\mu\partial^\mu\phi&=H\end{aligned} \tag{3.10}$$

which with H regarded as the gauge field and ϕ as the potential, can be viewed as an obvious generalisation of the Maxwell equations. The graph of β_0 is given in figure 1.

Now in the massive form (3.3), the only terms in $\mathbf{\Gamma}_\mu$ containing χ are, from (3.2), those corresponding to edges linking $\phi^{(s-3)}$, $\phi^{(s-4)},\ldots,\phi^{(0)}$ to other irreps, and we know that all of these fields decouple from the theory. Thus the s, $(s-1)$ and $(s-2)$ blocks of β_0 are the same as those of $\mathbf{\Gamma}_0$ for the massive theory, while lower-spin

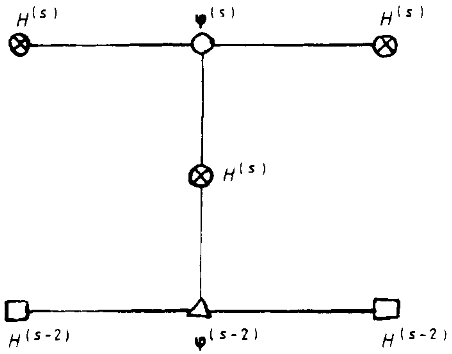

Figure 1. Graph of β_0, the $\mathscr{L}_p$ irreps involved being ($\otimes$) $H^{(s)} \sim \mathscr{D}[\frac{1}{2}(s+1), \frac{1}{2}(s-1)] \oplus \mathscr{D}[\frac{1}{2}(s-1), \frac{1}{2}(s+1)] \oplus \mathscr{D}[\frac{1}{2}(s-1), \frac{1}{2}(s-1)]$, ($\square$) $H^{(s-1)} \sim \mathscr{D}[\frac{1}{2}(s-1), \frac{1}{2}(s-3)] \oplus \mathscr{D}[\frac{1}{2}(s-3), \frac{1}{2}(s-1)]$, ($\bigcirc$) $\phi^{(s)} \sim \mathscr{D}(\frac{1}{2}s, \frac{1}{2}s)$, and ($\triangle$) $\phi^{(s-2)} \sim \mathscr{D}[\frac{1}{2}(s-2), \frac{1}{2}(s-2)]$, linked as indicated.

blocks all have the same graphical form as $\beta_0^{(s-2)}$ (and $\mathbf{\Gamma}_0^{(s-2)}$), but with different elements. The s block decomposition of the massless theory (3.3) and (3.4) is thus given in figure 2.

For the first three blocks (spin s, $(s-1$, $(s-2))$, the minimal polynomials must be identical to those of the corresponding $\mathbf{\Gamma}_0^{(s)}$, since these do not change. Thus

$$\beta_0^{(s)}(\beta_0^{(s)2}-1)=0 \tag{3.11a}$$

$$(\beta_0^{(s-1)})^3=0 \tag{3.11b}$$

$$(\beta_0^{(s-2)})^5=0 \tag{3.11c}$$

(these results are obtained directly from the graphs by the techniques outlined by Cox 1981). The remaining blocks of β_0 are no longer nilpotent in general, unlike the corresponding $\mathbf{\Gamma}_0$ spin blocks. To calculate the minimal polynomials for these blocks we need the appropriate matrix elements. These could be obtained by inspection and rearrangement of equations (3.6) and (3.7), but this is impracticable and unnecessary. We can instead use the standard results of Gel'Fand *et al* (1963) (also Cox 1974) to write down the general form of the elements for any relativistic and space reflection covariant theory derivable from a Lagrangian, with a graph as in figure 3 and use (3.11) to write down equations for these elements. It turns out that the equations (3.11) actually essentially determine uniquely the elements for all s blocks, the minimal polynomials of which may then be written down directly.

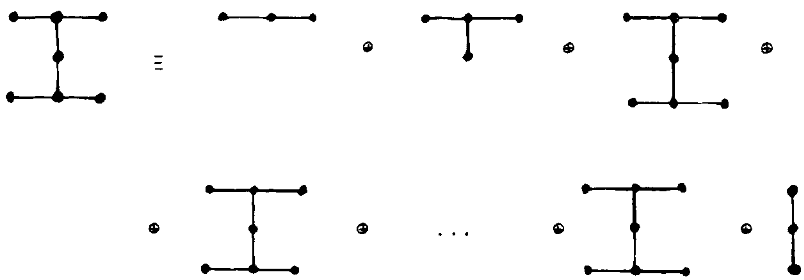

Figure 2. The s block decomposition of the massless theory (3.3) and (3.4).

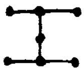

Figure 3. Uses the standard results of Gel'fand *et al* (also Cox) to write down the general form of the elements for any relativistic and space reflection covariant theory derivable from a Lagrangian with such a graph.

The graphs for the s, $(s-1)$ and $(s-2)$ blocks are given below, in figures 4, 5 and 6, with the corresponding matrix elements indicated, in the canonical representation (generator of rotations about z axis diagonal) of Gel'Fand *et al* (1963) and Cox (1974). The notation is

$$\rho(s, r) = |[(s+r+1)(s-r)]^{1/2}| \tag{3.12}$$

$$s_{ij} = \pm 1 \qquad \eta_i = \pm 1.$$

The C_{ij} are arbitrary complex numbers. The blocks correspond to the most general Lorentz and space reflection covariant theories derivable from a real invariant Lagrangian. They therefore include the Fronsdal equations as the massless limit of the Singh–Hagen equations.

The conditions (3.11) for these s blocks yield, from direct inspection of the graphs, the following conditions for the coefficients of the characteristic polynomials:

$$2\rho^2(s, -1)\eta_6 s_{56}|C_{56}|^2 = 1 \tag{3.13a}$$

$$2\rho^2(s-1, -1)\eta_6 s_{56}|C_{56}|^2 + s_{46}\rho^2(s-1, s)|C_{46}|^2 = 0 \tag{3.13b}$$

$$\begin{aligned} 2\rho^2(s-2, -1)\eta_6 s_{56}|C_{56}|^2 + \rho^2(s-2, s)s_{46}|C_{46}|^2 + \rho^2(s-2, s-1)s_{24}|C_{24}|^2 \\ + 2\rho^2(s-2, -1)\eta_2 s_{12}|C_{12}|^2 = 0 \end{aligned} \tag{3.13c}$$

$$\begin{aligned} 2\rho^2(s-2, -1)\eta_6 s_{56}|C_{56}|^2[2\rho^2(s-2, -1)\eta_2 s_{12}|C_{12}|^2 + \rho^2(s-2, s-1)s_{24}|C_{24}|^2] \\ + 2\rho^2(s-2, -1)\eta_2 s_{12}|C_{12}|^2\rho^2(s-2, s)s_{46}|C_{46}|^2 = 0. \end{aligned} \tag{3.13d}$$

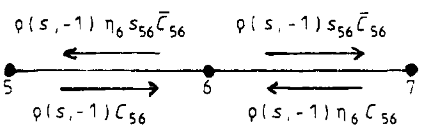

Figure 4. Graph for the s block.

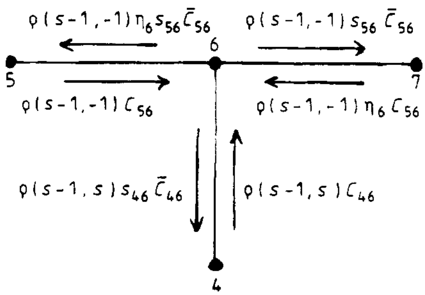

Figure 5. Graph for the $(s-1)$ block.

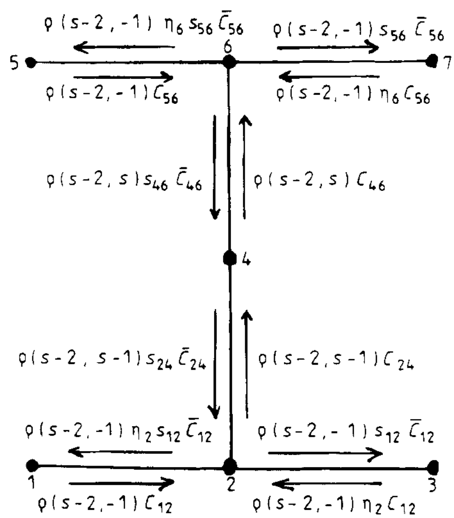

Figure 6. Graph for the $(s-2)$ block.

The first two equations yield

$$\eta_6 s_{56} = 1, \qquad s_{46} = -1, \qquad 2\rho^2(s,-1)|C_{56}|^2 = 1$$
$$|C_{46}|^2 = [\rho(s-1,-1)/\rho(s,-1)\rho(s-1,s)]^2$$

and the last two determine $|C_{12}|^2$ and $|C_{24}|^2$ in terms of $|C_{46}|^2$ and $|C_{56}|^2$. So the parameters C_{ij} are determined, up to arbitrary phases, by the conditions on the s, $(s-1)$ and $(s-2)$ blocks.

The $(s-r)$ block, $r \leqslant s-1$, has the general form shown in figure 7, and again direct inspection of the graph yields the minimal polynomial

$$\begin{aligned} m(\lambda) &= \lambda[\lambda^4 - C_2^{(r)}\lambda^2 + C_4^{(r)}] \\ &= [\lambda^4 - (2j_{56}^{(r)} + j_{46}^{(r)} + j_{24}^{(r)} + 2j_{12}^{(r)})\lambda^2 + [2j_{56}^{(r)}(2j_{12}^{(r)} + j_{24}^{(r)}) + 2j_{12}^{(r)}j_{46}^{(r)}]] \\ & \quad r = 3, 4, \ldots, (s-1) \end{aligned} \tag{3.14}$$

with the notation

$$\begin{aligned} j_{56}^{(r)} &= \rho^2(s-r,-1)\eta_6 s_{56}|C_{56}|^2 \\ j_{46}^{(r)} &= \rho^2(s-r,s)s_{46}|C_{46}|^2 \\ j_{24}^{(r)} &= \rho^2(s-r,s-1)s_{24}|C_{24}|^2 \\ j_{12}^{(r)} &= \rho^2(s-r,-1)\eta_2 s_{12}|C_{12}|^2 \qquad & r = 3, 4, \ldots, s. \end{aligned}$$

Note that the degree of nilpotency for these blocks is indeed one because from the graph the rank of each block is four (Cox 1981) and there are four non-zero eigenvalues, since in general $C_4^{(r)} \neq 0$, so the Jordan normal form of the zero eigenspace of each block must be the zero matrix.

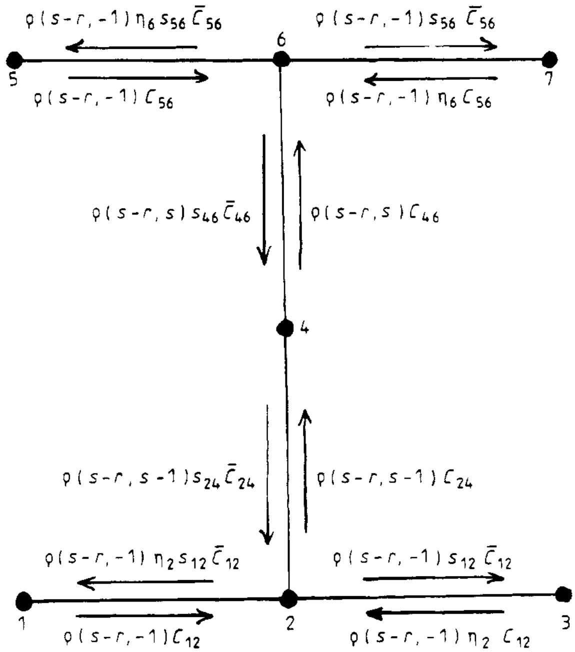

Figure 7. Graph of the $(s-r)$ block, $r \leq s-1$.

Finally, the zero block is given in figure 8, with minimal polynomial

$$m(\lambda) = \lambda[\lambda^2 - (j_{46}^{(s)} + j_{24}^{(s)})]. \tag{3.15}$$

Collecting the above results we find for the minimal polynomial of β_0:

$$\beta_0^5(\beta_0^2 - 1) \prod_{r=3}^{s-1} [\beta_0^4 - (2j_{56}^{(r)} + j_{46}^{(r)} + j_{24}^{(r)} + 2j_{12}^{(r)})\beta_0^2 + 2j_{56}^{(r)}(2j_{12}^{(r)} + j_{24}^{(r)}) + 2j_{12}^{(r)}j_{46}^{(r)}][\beta_0^2 - (j_{46}^{(s)} + j_{24}^{(s)})] = 0. \tag{3.16}$$

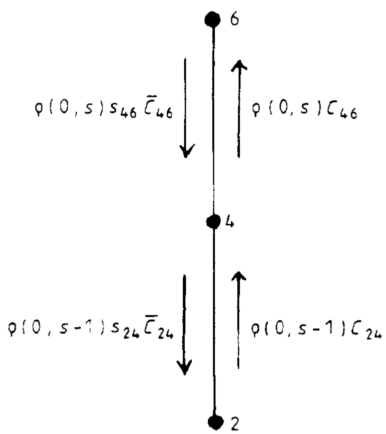

Figure 8. Graph of the zero block.

Covariantising and contracting with an appropriate number of p_μ, we find the operator $P = \beta p$ satisfies the minimal equation

$$P^5(P^2-1)\prod_{r=3}^{(s-1)}\{P^4-(2j_{56}^{(r)}+j_{46}^{(r)}+j_{24}^{(r)}+2j_{12}^{(r)})p^2P^2 \\ +(2j_{56}^{(r)}(2j_{12}^{(r)}+j_{24}^{(r)})+2j_{12}^{(r)}j_{46}^{(r)})p^4\}[P^2-(j_{46}^{(s)}+j_{24}^{(s)})p^2]=0. \qquad (3.17)$$

This is to be compared with the much simpler result for the corresponding spin-s massive theory:

$$P^{(2s-1)}(P^2-p^2)=0. \qquad (3.18)$$

While (3.17) is more complicated than (3.18), it in fact reflects a simpler structure for the β_0 matrix. In the massive case (3.18) all spin blocks except the s block are nilpotent and non-diagonalisable. For massive theories this is essentially a requirement of positivity for high-spin theories (Gel'Fand *et al* 1963). In the massless case (3.17), however, all spin blocks are diagonalisable except the $(s-1)$ and $(s-2)$ blocks, which are maximally nilpotent. Unfortunately, in the massless case there seems no direct connection between the β algebra and the mass–helicity spectrum. Whereas in the massive case one can achieve any required mass–spin spectra by appropriately selecting s blocks as nilpotent or having non-zero eigenvalues, when the mass matrix $\mathbf{M}$ is singular this procedure is no longer valid.

4. Conclusions

A great deal of work has been done on the theory of high-spin massive field equations written in the first-order Duffin–Kemmer form:

$$(\mathbf{L}_\mu\partial^\mu+\mathrm{i}\varkappa\mathbf{I})\psi=0, \qquad (4.1)$$

the emphasis being on the algebraic properties of the $\mathbf{L}_\mu$ matrices. However, very little is known about the analogous form for massless theories, which have mainly been studied in the conventional tensor–spinor form. In the massless case equation (4.1) must be replaced by an equation

$$(\beta_\mu\partial^\mu+\mathrm{i}\mathbf{M})\psi=0 \qquad (4.2)$$

where $\mathbf{M}$ is a singular matrix. The great value of the form (4.1) is the connection between the eigenvalue of $\mathbf{L}_0$ and the mass–spin spectra. This connection is not preserved in the massless form (4.2), and nothing is known about the algebraic properties of the β_μ. In this paper we have converted the high integer spin massless theory of Fronsdal to the form (4.2) and derived the form of the β algebra by calculating the minimal polynomials of the β_0 s blocks from known results for the massive Singh–Hagen theory. The resulting first-order theory is, apart from the β algebra, a direct generalisation of the Harish-Chandra massless theory for spin 1. For higher spin the β algebra differs markedly from the corresponding massive algebra. We have also noted that, as anticipated by Harish-Chandra, the range of antisymmetric tensor gauge field theories of recent interest can all be derived in a unified treatment from the Duffin–Kemmer equation with a singular mass matrix.

The advantage of (4.1) in the massive case is that it enables all covariant theories with a required mass–spin spectra to be studied systematically. In the conventional tensor–spinor formulation it is not easy to write down, *ab initio*, *all* possible theories with a required mass–spin spectra. The irreps involved and the connectivities between these required to yield the spectra desired are perhaps best seen using the s-block analysis of the $\mathbf{L}_0$ in the first instance and then converting to tensor form to supply the details (Cox 1982a, b). It was hoped that a greater understanding of the form (4.2) for massless theories might lead to similar benefits for massless equations, and further, by appropriate choice of β_μ, $\mathbf{M}$, it might be possible to obtain theories with any required spectra, including massive and massless particles, (Kwoh 1970). However, the complicated results for the β algebra of the massless Fronsdal equations are not encouraging in this respect.

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